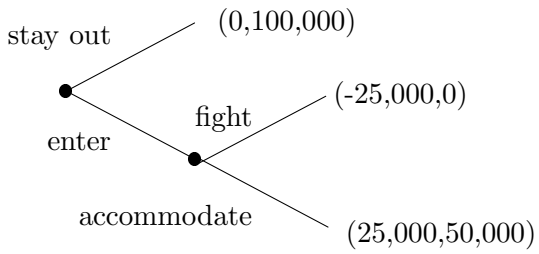
5

Answers and Hints

5.4 The Subway Entry Deterrence Game

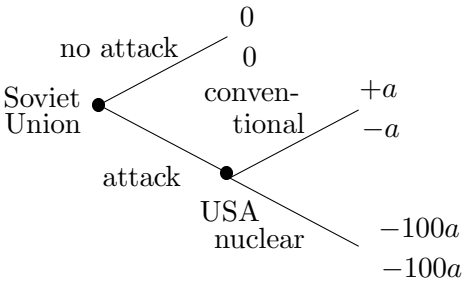
The game tree is:



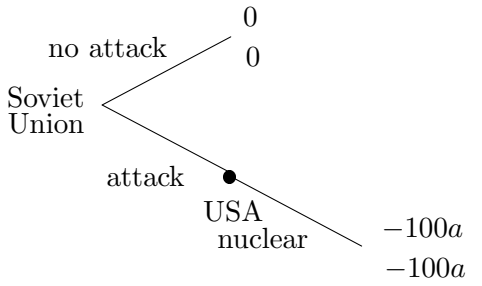
5.5 The Dr. Strangelove Game

1. The figure on the left is the extensive form of the game without precommitment;

2. The figure on the right is the extensive form of the game with precommitment;



USA Has no Credible Threat



USA Precommits to Nuclear Deterrent

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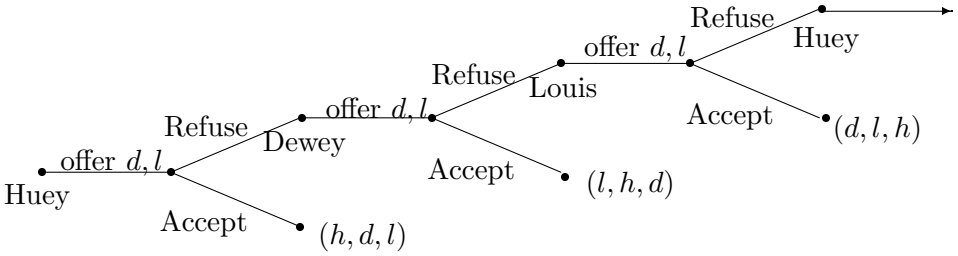
5.7 Huey, Dewey, and Louis Split a Dollar

Here is a game tree (written sideways), when the equilibrium shares are (h, d, l):



We work back the game tree (which is okay, since we are looking only for subgame perfect equilibria). At the second place where Huey gets to offer (at the right side of the game tree), the value of the game to Huey is h, since we assume a stationary equilibrium. Thus Louis must offer Huey at least δh where Louis gets to offer, to get Huey to accept. Similarly, Louis must offer Dewey at least δd at this node. Thus the value of the game where Louis gets to offer is $(1 - \delta h - \delta d)$.

When Dewey gets to offer, he must offer Louis at least δ times what Louis gets when it is Louis' turn to offer, to get Louis to accept. This amount is just $\delta(1-\delta h-\delta d)$. Similarly, he must offer Huey $\delta^2 h$ to accept, since Huey gets δh when it is Louis' turn to offer. Thus Dewey gets

$$1 - \delta(1 - \delta h - \delta d) - \delta^2 h = 1 - \delta(1 - \delta d)$$

when it is his turn to offer.

Now Huey, on his first turn to offer, must offer Dewey δ times what Dewey can get when it is Dewey's turn to offer, or $\delta(1 - \delta(1 - \delta d))$. But then we must have

$$d = \delta(1 - \delta(1 - \delta d)).$$

Solving this equation for d, we find

$$d = \frac{\delta}{1 + \delta + \delta^2}.$$

Moreover, Huey must offer Louis δ times what Dewey would offer Louis in the next period, or $\delta^2(1-\delta h-\delta d)$. Thus Huey offers Dewey and Louis together

$$\delta(1 - \delta(1 - \delta d)) + \delta^2(1 - \delta h - \delta d) = \delta - \delta^3 h,$$

so Huey gets $1 - \delta + \delta^3 h$ and this must equal h. Solving, we get

$$h = \frac{1}{1 + \delta + \delta^2},$$

so we must have

$$l = 1 - d - h = \frac{\delta^2}{1 + \delta + \delta^2},$$

which is the solution to the problem.

Note that there is a simpler way to solve the problem, just using the fact that the solution is symmetric: we must have $d = \delta h$ and $l = \delta d$, from which the result follows. This does not make clear, however, where subgame perfection comes in.

5.10 Cooperation in an Overlapping Generations Economy

Part (d): Let T^* be the age of veneration. Then for a member of age $t \leq T^*$, the gains from cooperating are $(T - t + 1)nT^* - (T^* - t + 1)\alpha$, and the gains from defecting are $nT^* - 1$. Thus the net gains from cooperating are

$$f(t, T^*) = (T - t)nT^* - (T^* - t + 1)\alpha + 1.$$

Then T^* supports a Nash subgame perfect equilibrium of the desired form if and only if $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T*$. In particular, we must have $f(T^*, T^*) \geq 0$. But $f(t, t) = (T - t)nt - (\alpha - 1)$ is a parabola with a maximum at $t = T/2$, and $f(T/2, T/2) = nT^2/4 - (\alpha - 1)$. Since this must be nonnegative, we see that

$$T^2 \geq \frac{4(\alpha - 1)}{n} \tag{5.1}$$

is necessary.

Now suppose (5.1) holds, and choose T^* such that $f(T^*, T^*) \geq 0$. Since $f_t(t, T^*) = -nT^* + \alpha$, if $\alpha \leq nT^*$ then f is decreasing in t, so $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T^*$. If $\alpha > nT^*$, $f(t, T^*)$ is increasing in t, so we must ensure that $f(1, T^*) \geq 0$. But $f(1, T^*) = T^*(n(T - 1) - \alpha) + 1$, which is strictly positive by assumption. Thus (5.1) is sufficient.

Part (e): The total utility from the public good for a member is $T^*(nT - \alpha)$, which is an increasing function of T^*.

5.12 The Finitely Repeated Prisoner's Dilemma: the Fine Points

(a) A string of k defections followed by a return to cooperation costs the player $k+1-(a-b)$, which is strictly positive for $k \geq 1$. To see this, we can clearly assume we are at round 1, since the argument is the same further down the line. Cooperating until the last round, then defecting, pays $n-1+a$, while defecting for k rounds and then cooperating returns $a-b+(n-k-2)+a$ (you get a on the first round, you get nothing for $k-1$ rounds, then you lose b when you return to cooperate, then you get 1 for $n-k-2$ rounds, then you defect on the last round and get a).

(b) By preempting, player 1 is revealed to be a best responder (reciprocators never preempt). The statement is clearly true if player 2 is a reciprocator. If player 2 is a best responder, and if he does not defect, player 1 would know he is best responder. Since both players would now know they are not reciprocators, they would both defect forever. Meanwhile player 2 would lose b on this round by cooperating. So player 2 should defect in response to preemption. Moreover, if later player 2 cooperated while player 1 continued to defect, player 1 would again know that player 2 is a best responder, and so both would defect forever. The assertion follows.

(c) The best case scenario is that when player 1 returns to C, player 2 does so as well. But in (a), we saw that player 1 can't gain from such a move.

(d) We know that player 1 cannot gain by preempting at stage $k-1$ and returning to C at some later stage. Thus by preempting at $k-1$, player 1 will defect thereafter in an optimal strategy. The payoff to this is a. But by waiting until next period to defect, he gets 1 now and ϵa next period. But by assumption $(1-\epsilon)a < 1$, so $1+\epsilon a > a$, so waiting dominates preempting.

(e) We now know that a best responder should either cooperate until the last round, or preempt exactly when he expects his partner will preempt if his partner is a best responder. The total payoff to the latter strategy is $k-1+\epsilon a$ if he thinks a best responder will switch on round k, and the payoff to the former strategy is $(k-1-b)(1-\epsilon)+(n-1+a)\epsilon$. The preempt strategy is thus

superior precisely when

$$k > n - \frac{b(1-\epsilon)}{\epsilon},$$

and then only if the player believes a best responder will preempt on round k.

(f) This follows from the fact that a best responder will never preempt.

5.15 Selten's Horse

If Walter's information set is not reached with positive probability, an equilibrium must have the form $(A, a, p_\lambda\lambda + (1-p_\lambda)\rho)$. Franz's payoff to D is $4p_\lambda + (1-p_\lambda)$ which must be at most 3, so $p_\lambda \leq 2/3$. Gustav's payoff to d is $5p_\lambda + 2(1-p_\lambda)$ which must be at most 3, so $p_\lambda \leq 1/3$. This gives us the set M of equilibria.

Now assume Walter's information set is reached. If Walter chooses the pure strategy λ, either Gustav must not get to choose, or Gustav must choose d. If Gustav does not get to choose, Franz must choose D, with payoffs (4,4,4). If Gustav plays a with probability p_a, the payoff to A is $3p_a + 5(1-p_a)$, which must be at most 4, so $p_a \geq 1/2$. This gives us the set N of equilibria. If Gustav does get to choose, Gustav chooses d, so Franz must randomize or else Walter would not choose λ. But the payoff to A for Walter is 5, and the payoff to D is 4, so Franz will not randomize. Thus there are no Nash equilibria in this category.

If Walter's information set is reached and Walter chooses pure strategy ρ, either Gustav does not get to choose, or Gustav chooses a. If Gustav does not get to choose, Franz chooses D, which is clearly inferior to A, no matter what Gustav's strategy is. If Gustav does choose, Gustav chooses a, so $p_A = 1$, which contradicts the assumption that Walter's information set is reached with positive probability.

The remaining case is that Walter's information set is reached with positive probability and Walter randomizes. The payoff to λ is $4p_{3l}$, where p_{3l} is the probability of being at $3l$, and the payoff to ρ is $p_{3l} + 2(1-p_{3l})$, and since these must be equal, we have $p_{3l} = 2/5$. But clearly $p_{3l} = (1-p_A)/(1-p_A+p_A(1-p_a))$. This implies $p_A(5-2p_a) = 3$. Clearly p_A cannot then be zero or one, so Franz randomizes, which implies

$$4p_\lambda + (1-p_\lambda) = 3p_a + (1-p_a)(5p_\lambda + 2(1-p_\lambda)),$$

or $p_a(3p_\lambda - 1) = 1$. If Gustav randomizes, we must have $3 = 5p_\lambda + 2(1 - p_\lambda)$, so $p_\lambda = 1/3$, which is impossible. Thus $p_a = 1$ and $p_\lambda = 1/3$, which is again impossible. So this case is impossible.

5.16 Trembling Hand Perfection

(a) A straightforward calculation yields

$$p_{3l} = \frac{\epsilon_1 + (1 - 2\epsilon_1)(1 - p_A)}{\epsilon_1 + (1 - 2\epsilon_1)(1 - p_A) + (\epsilon_1 + (1 - 2\epsilon_1)p_A)(\epsilon_2 + (1 - 2\epsilon_2)(1 - p_a))}.$$

(b) If $p_A < 1$, then p_{3l} approaches

$$p_{3l}^{\infty} = \frac{(1 - p_A)}{(1 - p_A) + p_A(1 - p_a)}$$

for small ϵ_1, ϵ_2. Suppose $p_A < 1$ Then $p_\lambda = 1$ for small ϵ_1, ϵ_2 if $p_{3l}^{\infty} \geq 2/5$, which reduces to $3 \geq p_A(5 - 2p_a)$. In this situation Gustav chooses d, so $p_a = 0$. The payoff of A to Walter is 5 and the payoff to D is 4, so $p_A = 1$, which is a contradiction. We next try $p_\lambda = 0$ for small ϵ_1, ϵ_2, can be true if $3 \leq p_A(5 - 2p_a)$. Now $p_a = 1$, so $p_A = 1$, a contradiction. The only alternative if $p_A < 1$ is that Walter randomizes for small ϵ_1, ϵ_2, which means that $3 = p_A(5 - 2p_a)$. It follows that Franz randomizes as well, and the condition that the payoffs to A and D are equal for Franz reduces to $p_a(3p_\lambda - 1) = 1$. This implies $p_\lambda > 1/3$, so the payoff to d is greater than $5(1/3) + 2(2/3) = 3$, so $p_a = 0$, which is a contradiction.

We conclude that for sufficiently small ϵ_1, ϵ_2, $p_A = 1$, which implies

$$p_{3l} = \frac{\epsilon_1}{\epsilon_1 + (1 - \epsilon_1)(\epsilon_2 + (1 - 2\epsilon_2)(1 - p_a))}.$$

It is clear that in this case Walter cannot choose $p_\lambda = 1$ since then $p_a = 0 \rightarrow p_\lambda = 0$, a contradiction. If $p_\lambda = 0$ then $p_a = 1$, so

$$p_{3l} = \frac{\epsilon_1}{\epsilon_1 + (1 - \epsilon_1)\epsilon_2}.$$

This is must be a local best response for Walter, which is the case so long at $p_{3l} \geq 2/5$, which is equivalent to $3\epsilon_1 \geq 2(1 - \epsilon_1)e_2$. If this inequality fails, then Walter must randomize, which means $p_{3l} = 2/5$, and given ϵ_1 and ϵ_2, this uniquely determines $p_a < 1$. In this case as $\epsilon_1, \epsilon_2 \rightarrow 0$ while maintaining $3\epsilon_1 < (1 - \epsilon_1)e_2$, $p_a \rightarrow 1$, so in all cases for small ϵ_1, ϵ_2, the equilibrium is near the perfect equilibrium.